

A Note on Chromatic Sum

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Abstract

The chromatic sum $\Sigma(G)$ of a graph G is the smallest sum of colors among of proper coloring with the natural number. In this paper, we introduce a necessary condition for the existence of graph homomorphisms. Also, we present $\Sigma(G) < \chi_f(G)|G|$ for every graph G .

Key words: chromatic sum, graph homomorphism, Fractional chromatic number.

Subject classification: 05C .

1 Introduction and Preliminaries

We consider finite undirected graphs with no loops and multiple edges and use [4] for the notions and notations not defined here. Let G be a graph and c be a proper coloring of it, define $\Sigma_c(G) = \sum_{v \in V(G)} c(v)$. The vertex-chromatic sum of G , denoted by $\Sigma(G)$, is defined as $\min\{\Sigma_c(G) \mid c \text{ is a proper coloring of } G\}$. The vertex-strength of G denoted by $s(G)$, or briefly by s , is the smallest number s such that there is a proper coloring c with s colors where $\Sigma_c(G) = \Sigma(G)$. Clearly, $s(G) \geq \chi(G)$ and equality does not always hold. In fact, for every positive integer k , almost all trees satisfy $s > k$; see [7]. Chromatic sum has been investigated in literature [1, 2, 3, 5, 6, 7, 10].

In [10], Thomassen et al. obtained several bounds for chromatic sum for general graphs. The first is a rather natural result of an application of a greedy algorithm: $\Sigma(G) \leq n + e$, where n and e are the number of vertices and edges of G , respectively. Also, they presented an upper and lower limit for the chromatic sum in terms of e . They showed that $\sqrt{8e} \leq \Sigma(G) \leq \frac{3}{2}(e + 1)$ and these bounds are sharp.

Let G and H be two graphs. A homomorphism σ from a graph G to a graph H is a map $\sigma : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. The set of all homomorphisms from G to H is denoted by $\text{Hom}(G, H)$. An isomorphism of G to H is a homomorphism $f : G \rightarrow H$ which is a vertex and edge bijective homomorphism. An isomorphism $f : G \rightarrow G$ is called an automorphism of G , and the set of all automorphism of G is denoted by $\text{Aut}(G)$.

Suppose $m \geq 2n$ are positive integers. We denote by $[m]$ the set $\{1, 2, \dots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n -subsets of $[m]$. The Kneser graph $KG(m, n)$

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²This paper is partially supported by Shahid Beheshti University.

has vertex set $\binom{[m]}{n}$, in which $A \sim B$ if and only if $A \cap B = \emptyset$. The graph $KG(5, 2)$ is named Petersen graph that is denoted by P . It was conjectured by Kneser in 1955 and proved by Lovász [8] in 1978 that $\chi(KG(m, n)) = m - 2n + 2$.

The fractional chromatic number of a graph G , denoted by $\chi_f(G)$, is the infimum of the ratios $\frac{m}{n}$ such that there is a homomorphism from G to $KG(m, n)$. It is known [9] that the infimum in the definition can be attained, and hence can be replaced by the minimum. It is easy to see $\chi_f(G) \leq \chi(G)$. On the other hand, the ratio $\frac{\chi(G)}{\chi_f(G)}$ can be arbitrary large, see [9].

In next section we present a necessary condition for existence of graph homomorphisms in terms of chromatic sum. Next, we introduce an upper bound for chromatic sum based on fractional chromatic number.

2 Graph Homomorphism and Chromatic Sum

Graph homomorphism is a fundamental concept in graph theory, where it is related to many important concepts and problems in the field. It is well-known that in general it is a hard problem to decide whether there exists a homomorphism from a given graph G to a given graph H , and consequently, it is interesting to obtain necessary conditions for the existence of such mappings. In this regard, we have the following theorem.

Theorem 1. *Let G and H be two graphs such that H is a vertex transitive graph. If $\sigma : G \rightarrow H$ is a homomorphism, then*

$$\frac{\Sigma(G)}{|G|} \leq \frac{\Sigma(H)}{|H|}.$$

Proof. Let $\text{Aut}(H) = \{f_1, f_2, \dots, f_t\}$ and $\tilde{G} = \bigcup_{i=1}^t G_i$ that G_i is an isomorphic copy of G . Define $\tilde{\sigma} : \tilde{G} \rightarrow H$ such that its restriction to G_i is $f_i \circ \sigma$. Since H is a vertex transitive graph, one can easily show that for every $v \in V(H)$, $|\tilde{\sigma}^{-1}(v)| = t \frac{|G|}{|H|}$ and it is independent of v . Now, suppose c is a proper coloring of H such that $\Sigma_c(H) = \Sigma(H)$. For any vertex $v \in V(\tilde{G})$, set $\tilde{c}(v) = c(\tilde{\sigma}(v))$. Obviously, \tilde{c} is a proper coloring of \tilde{G} and also $\Sigma_{\tilde{c}}(\tilde{G}) = \frac{t|G|}{|H|} \times \Sigma(H)$. Therefore, there is an i such that $\Sigma_{\tilde{c}|_{G_i}}(G_i) \leq \frac{|G|}{|H|} \times \Sigma(H)$ and since $G = G_i$, $\Sigma(G) \leq \frac{|G|}{|H|} \times \Sigma(H)$ which is the desired conclusion. ■

Theorem 1 provides a necessary condition for the existence of graph homomorphisms. Here we show that The Petersen graph P has the same chromatic number and circular chromatic number. One can check that $\Sigma(P) = 19$ and $\Sigma(K_{\frac{8}{3}}) = 15$. Therefore, as an application of the previous theorem, there is no homomorphism from P to $K_{\frac{8}{3}}$.

It is well-known that the chromatic sum is an NP-complete problem[7]. In this regard, finding upper and lower bounds for chromatic sum is useful. It was shown in [3] that $\Sigma(G) \leq (\frac{\chi(G)+1}{2})|G|$. Since $\Sigma(K_n) = \frac{n(n+1)}{2}$, if we set $H = K_{\chi(G)}$, then Theorem 1 implies this bound. Here we obtain an upper bound for the chromatic sum in terms of fractional chromatic number.

For an independent set S in a graph G the following inequality is an immediate consequence of the definition of the chromatic sum ([2]),

$$\Sigma(G) \leq |G| + \Sigma(G \setminus S). \quad (1)$$

Theorem 2. *For every graph G , we have*

$$\Sigma(G) < \chi_f(G)|G|.$$

Proof. Assume that $\chi_f(G) = \frac{m}{n}$ and $\text{Hom}(G, KG(m, n)) \neq \emptyset$. In view of equation 1, we have $\Sigma(KG(m, n)) \leq \binom{m}{n} + \Sigma(KG(m-1, n))$. Hence $\Sigma(KG(m, n)) \leq \sum_{i=0}^{m-2n-1} \binom{m-i}{n} + \Sigma(KG(2n, n))$. On the other hand, $\sum_{i=0}^{m-2n-1} \binom{m-i}{n} = \binom{m+1}{n+1} - \binom{2n+1}{n+1}$ and $\Sigma(KG(2n, n)) = \frac{3}{2} \binom{2n}{n}$. Therefore, $\Sigma(KG(m, n)) \leq \binom{m+1}{n+1} - \binom{n-1}{2n+2} \binom{2n}{n}$. Now, since $\text{Hom}(G, KG(m, n)) \neq \emptyset$, Theorem 1 implies that

$$\Sigma(G) \leq \left(\frac{m+1}{n+1} - \binom{n-1}{2n+2} \frac{\binom{2n}{n}}{\binom{m}{n}} \right) |G|.$$

Furthermore, $\frac{m+1}{n+1} - \binom{n-1}{2n+2} \frac{\binom{2n}{n}}{\binom{m}{n}} \leq \frac{m+1}{n+1} < \frac{m}{n} = \chi_f(G)$, as desired. \blacksquare

In particular, if G is a vertex transitive graph, $\chi_f(G) = \frac{|G|}{\alpha(G)}$ and hence $\Sigma(G) < \frac{|G|^2}{\alpha(G)}$. Furthermore, $e(G) = \frac{\Delta(G)|G|}{2}$. If $\chi_f(G) \leq \frac{3}{4}\Delta(G)$, then $\chi_f(G)|G| < \frac{3}{2}(e(G) + 1)$. Therefore, the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G) + 1)$ (see [10]).

On the other hand, in view of Theorem 1 we have

$$\Sigma(G) \geq \frac{\omega(G) + 1}{2} |G|$$

where G is a vertex transitive graph and $\omega(G)$ is the size of the largest clique in it.

Also, It is a known result that the ratio $\frac{\chi(G)}{\chi_f(G)}$ can be arbitrary large (see [9]). Let $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ such that $\frac{\chi(G_n)}{\chi_f(G_n)} \rightarrow \infty$. We can assume that G_n is critical for all n (G is critical if $\chi(G \setminus v) < \chi(G)$ for every $v \in V(G)$). Thus, $e(G_n) \geq \frac{|G_n|(\chi(G_n)-1)}{2}$ and we also have $\frac{\frac{3}{2}(e(G_n)+1)}{\chi_f(G_n)|G_n|} \rightarrow \infty$. It means the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G) + 1)$ for the graphs in \mathcal{G} .

In Theorem 2 we used an upper bound of $\Sigma(KG(m, n))$, but we do not know the exact value of $\Sigma(KG(m, n))$. The improvement of this upper bound yields an improvement in Theorem 2.

Problem 1 *What is the exact value of $\Sigma(KG(m, n))$? Is it true that $\Sigma(KG(m, n)) = \binom{m}{n} \frac{m+1}{n+1} - \binom{n-1}{2n+2} \frac{\binom{2n}{n}}{\binom{m}{n}}$?*

Acknowledgment

We would like to thank Hossein Hajiabolhassan and Moharam Nejad Iradmusa for their useful comments.

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